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A Constructive Path to the Hyperreal Numbers from Sequences of Naturals (Draft 17)

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The objective of this section is to build the foundations of the construction of the hypernatural numbers. This will be accomplished in a generalized setting: letting $\mathbb{X}$ be

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an algebraic structure defined on the set X, the goal is to construct the hyper-structure ${}^*\mathbb{X}$ which preserves the key operational, existential, and relational properties of $\mathbb{X}$ while including unlimited values. As an intermediate extension of $\mathbb{X}$, we will first develop a structure on the set

$$\mathrm{X}^{\mathrm{I}} = \{f \mid f : \mathrm{I} \to \mathrm{X}\}$$

as a generalization of the ordered pairs and sequences used in the standard construction of the integers, rationals, and reals. Since it will often be useful to specify these functions directly from images, the notations $\langle f(i)\rangle_{i\in\mathrm{I}}$ and $\langle f_i\rangle_{i\in\mathrm{I}}$ will often be used. Further, for the sake of simplicity when the context permits, the index will be omitted with the understanding that $i \in \mathrm{I}$.

Any operation defined on X is naturally extended to X^{I}. In general, given the operation $\star$ on X, $\star^{\mathrm{I}}$ on X^{I} is given by

$$(f \star^{\mathrm{I}} g)(i) = f(i) \star g(i)$$

for any $f, g \in \mathrm{X}^{\mathrm{I}}$ and all $i \in \mathrm{I}$. The operational and existential properties on X^{I} are readily verified and are thus left to the reader. It is far more labourious to extend relations on X to X^{I}. In particular, since the key case we are interested in is when $\mathbb{X} = (\mathbb{N}, +, \cdot)$, we will need to extend equivalence and ordering relations from X to X^{I}. Further paralleling the standard constructions of the naturals, integers, and reals, ${}^*\mathrm{X}$ is defined as $\mathrm{X}^{\mathrm{I}}/\mathrm{E}^{\mathrm{I}}$, where E^{I} the extension of the equivalence relation E on X. Now with this described, stating the objectives of this construction can be made more rigourous.

1. ${}^*\mathbb{X}$ extends $\mathbb{X}$.
   (a) Any key operational (associative, commutative, distributive), existential (identities, inverses), or relational (total ordering compatible with operations) properties true in $\mathbb{X}$ are also true in ${}^*\mathbb{X}$
   (b) $\mathbb{X}$ is isomorphic to a subset of ${}^*\mathbb{X}$. Equivalently, there exists an injective homomorphism $\phi : \mathrm{X} \to {}^*\mathrm{X}$.
2. There exists a nonstandard element in ${}^*\mathrm{X}$. That is, there exists $\alpha \in {}^*\mathrm{X}$ such that $\alpha \neq \phi(x)$ for all $x \in \mathrm{X}$.
3. One of these nonstandard elements is unlimited (greater than all $\phi(x)$ for $x \in \mathrm{X}$) or infinitesimal (less than all $\phi(x)$ for positive $x \in \mathrm{X}$).

Given a relation R on X, the extended relation ${}^*\mathrm{R}$ will be based on the indices in I for which R holds. As such, it is convenient to introduce the following notation.

Definition 3.1. Given $f, g \in \mathrm{X}^{\mathrm{I}}$,

$$[\![f\mathrm{R}g]\!] = \{i \in \mathrm{I} : f_i \mathrm{R} g_i\}$$

Definition 3.2. Given $\mathscr{F} \subseteq \mathscr{P}(\mathrm{I})$, a relation R on X is extended to a relation $^*\mathrm{R}$ on X^I as follows:

$$f\,^*\mathrm{R}g \iff [\![f\mathrm{R}g]\!] \in \mathscr{F}$$

for $f, g \in \mathrm{X}^\mathrm{I}$.

With this, many properties of $^*\mathrm{R}$ and thereby of $^*\mathbb{X}$ are direct results of properties of the set $\mathscr{F}$. For example, if $\mathscr{F} = \mathscr{P}(\mathrm{I})$ then $^*\mathrm{X}$ would only consist of a single equivalence class since any two $f, g \in \mathrm{X}^\mathrm{I}$ would be equivalent. To identity which properties $\mathscr{F}$ must have so that $^*\mathbb{X}$ is the desired construction, we shall pursue the proofs for extending equivalence and ordering relations. Let E be an equivalence relation on X. Then,

1. Reflexive. For $f\,^*\mathrm{E}f$, it must be that $[\![f\mathrm{E}f]\!] = \mathrm{I} \in \mathscr{F}$.

2. Symmetric. $f\,^*\mathrm{E}g \iff [\![f\mathrm{E}g]\!] \in \mathscr{F} \iff [\![g\mathrm{E}f]\!] \in \mathscr{F} \iff g\,^*\mathrm{E}f$ is already true for $\mathscr{F}$, so nothing more is required.

3. Transitive. Suppose $f\,^*\mathrm{E}g$ and $g\,^*\mathrm{E}h$ so that $[\![f\mathrm{E}g]\!] \in \mathscr{F}$ and $[\![g\mathrm{E}h]\!] \in \mathscr{F}$. Observe then that

$$[\![f\mathrm{E}h]\!] \supseteq [\![f\mathrm{E}g]\!] \cap [\![g\mathrm{E}h]\!]$$

This is true if $\mathscr{F}$ is closed under pairs of intersections and supersets.

Additionally, as mentioned previously, if $\mathrm{F} = \mathscr{P}(\mathrm{I})$ then $^*\mathrm{E}$ would trivially extend $\mathbb{X}$. As such, we also require that $\mathrm{F} \neq \mathscr{P}(\mathrm{I})$. With this, the following structure on $\mathscr{F}$ is imposed so that equivalence relations are nontrivially extended.

Definition 3.3. The nonempty family $\mathscr{F} \subset \mathscr{P}(\mathrm{I})$ is a filter if it has the

1. Superset property. $\forall \mathrm{A} \in \mathscr{F}, \forall \mathrm{B} \subseteq \mathrm{I}, \mathrm{A} \subseteq \mathrm{B} \implies \mathrm{B} \in \mathscr{F}$

2. Intersection property. $\forall \mathrm{A}, \mathrm{B} \in \mathscr{F}, \mathrm{A} \cap \mathrm{B} \in \mathscr{F}$

As required, I is also always in $\mathscr{F}$ since I is a superset of any element in $\mathscr{F}$. Additionally, observe that $\emptyset \notin \mathscr{F}$, as otherwise $\mathscr{F} = \mathscr{P}(\mathrm{I})$ by the superset property. Lastly, note that by induction, the intersection property is equivalent to any finite intersection of sets from the filter is also being in the filter.

Further, consider what properties $\mathscr{F}$ must have for $^*\mathrm{T}$ to be a total ordering if T is.

1. Reflexive, Transitive. Same proofs as for equivalence relations, so $\mathscr{F}$ being a filter satisfies these.

2. Antisymmetric. Suppose that $f\,^*\mathrm{T}g$ and $g\,^*\mathrm{T}f$ so that $[\![f\mathrm{T}g]\!], [\![g\mathrm{T}f]\!] \in \mathscr{F}$. If $\mathscr{F}$ is a filter then

$$[\![f\mathrm{T}g]\!] \cap [\![g\mathrm{T}f]\!] = [\![f\mathrm{E}g]\!] \in \mathscr{F}$$

3. Total. Assume that $f \neg {}^*\mathrm{T} g$ so that $[\![f\mathrm{T}g]\!] \notin \mathscr{F}$. Then,

$$\mathrm{I} \setminus [\![f\mathrm{T}g]\!] = [\![g\mathrm{T}f]\!] \cap (\mathrm{I} \setminus [\![f\mathrm{E}g]\!]) \subseteq [\![g\mathrm{T}f]\!]$$

   Therefore, if it were the case that $[\![f\mathrm{T}g]\!] \notin \mathscr{F} \implies \mathrm{I} \setminus [\![f\mathrm{T}g]\!] \in \mathscr{F}$, then ${}^*\mathrm{T}$ would be total.

Since $\mathscr{F}$ is a filter, only this final property must be imposed to ensure that ${}^*\mathrm{T}$ is a total ordering.

Definition 3.4. If $\mathscr{F}$ is a filter, it is said to be an ultrafilter if for all $\mathrm{A} \subseteq \mathrm{I}$, either $\mathrm{A} \in \mathscr{F}$ or $\mathrm{I} \setminus \mathrm{A} \in \mathscr{F}$.

Note that exclusively either A or $\mathrm{I} \setminus \mathrm{A}$ is in $\mathscr{F}$, since $\mathrm{A} \cap (\mathrm{I} \setminus \mathrm{A}) = \emptyset$ cannot be an element of any filter. To summarize, if $\mathscr{F}$ is an ultrafilter and if E and T are an equivalence and ordering relation on X respectively, then ${}^*\mathrm{E}$ and ${}^*\mathrm{T}$ are an equivalence and ordering relation on ${}^*\mathrm{X}$ respectively.

However, recalling the two objectives of the construction of ${}^*\mathbb{X}$, it becomes clear that not all ultrafilters will satisfy these goals. In particular, ultrafilters containing a singleton cannot be used to sufficiently extend $\mathbb{X}$. Such an ultrafilter would imply that any two functions are equivalent only if the elements in their ith terms are equal. As the reader can verify in further detail, this would imply that $\mathbb{X} \simeq {}^*\mathbb{X}$. Since $\mathscr{F}$ is an ultrafilter, this is equivalent to requiring that $\mathscr{F}$ contains no finite sets. To see this, let $\mathrm{A} = \{a_1, \dots, a_n\} \in \mathscr{F}$. Then, it must be that $\{a_k\} \in \mathscr{F}$ for some $1 \le k \le n$ since otherwise $\mathrm{I} \setminus \{a_k\} \in \mathscr{F}$ so thus

$$\begin{aligned}
\bigcap_{k=1}^{n} \mathrm{I} \setminus \{a_k\} &= \mathrm{I} \setminus \bigcup_{k=1}^{n} \{a_k\} \\
&= \mathrm{I} \setminus \mathrm{A} \\
&\implies \mathrm{A} \cap (\mathrm{I} \setminus \mathrm{A}) = \emptyset \in \mathscr{F}
\end{aligned}$$

This motivates the following definition.

Definition 3.5. Let $\mathscr{F}$ be an ultrafilter. $\mathscr{F}$ is said to be nonprincipal if for every $\mathrm{A} \in \mathscr{F}$ implies that A is infinite.

With the numerous restrictions now imposed on $\mathscr{F}$, it is not at all obvious that such a set can exist. To demonstrate that a nonprincipal ultrafilter does exist on an arbitrary unlimited set I, first consider the generation of filters.

Definition 3.6. Let $\mathscr{H} \subseteq \mathscr{P}(\mathrm{I})$. $\mathscr{H}$ is said to have the finite intersection property (or fip) if for any $n \in \mathbb{N}$ and $\mathrm{B}_1, \dots, \mathrm{B}_n \in \mathscr{H}$, $\mathrm{B}_1 \cap \cdots \cap \mathrm{B}_n \neq \emptyset$

Definition 3.7. Let $\mathscr{H} \subseteq \mathscr{P}(I)$ have the finite intersection property. Then, the filter generated by $\mathscr{H}$ is

$$\mathscr{F}^{\mathscr{H}} = \{A \subseteq I : A \supseteq B_1 \cap \cdots \cap B_n \text{ for any } n \in \mathbb{N}, B_1, \ldots, B_n \in \mathscr{H}\}$$

and is also the smallest filter containing $\mathscr{H}$.

To prove that $\mathscr{F}^{\mathscr{H}}$ is the smallest filter containing $\mathscr{H}$, it suffices to show that $\mathscr{F}^{\mathscr{H}}$ is a filter that contains $\mathscr{H}$ and if $\mathscr{F}$ is any filter that contains $\mathscr{H}$ then $\mathscr{F}^{\mathscr{H}} \subseteq \mathscr{F}$. Indeed, $\mathscr{F}^{\mathscr{H}}$ is a filter since

1. Superset property. For any $B \supseteq A$ where $A \in \mathscr{F}^{\mathscr{H}}$, A is the superset of a finite intersection of sets from $\mathscr{H}$, so thus B is as well and so $B \in \mathscr{F}^{\mathscr{H}}$.

2. Intersection property. For $A, B \in \mathscr{F}^{\mathscr{H}}$, $A \supseteq A_1 \cap \cdots \cap A_n$ and $B \supseteq B_1 \cap \cdots \cap B_m$ for some $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathscr{H}$. Therefore, $A \cap B \supseteq A_1 \cap \cdots \cap A_n \cap B_1 \cap \cdots \cap B_n$, so thus $A \cap B \in \mathscr{F}^{\mathscr{H}}$.

3. And $\emptyset \notin \mathscr{F}^{\mathscr{H}}$ since it would only occur when there is some empty finite intersection of $\mathscr{H}$, which is avoided by the assumption of the finite intersection property.

Further, $\mathscr{H} \subseteq \mathscr{F}^{\mathscr{H}}$ since every $H \in \mathscr{H}$ is a superset of itself. Lastly, if $\mathscr{F}$ is a filter where $\mathscr{H} \subseteq \mathscr{F}$ then any superset of any finite intersection of sets from $\mathscr{H}$ is also included in $\mathscr{F}$, so $\mathscr{F}^{\mathscr{H}} \subseteq \mathscr{F}$. This verifies that $\mathscr{F}^{\mathscr{H}}$ is the smallest filter that contains $\mathscr{H}$.

Filter generation provides a useful tool for extending filters. Given a filter $\mathscr{F}$ and a set $A \subseteq I$, it is not generally true that $\mathscr{F} \cup \{A\}$ is a filter. However, $\mathscr{F}^{\mathscr{F} \cup \{A\}}$ is the smallest filter containing both $\mathscr{F}$ and A. With the ability to create larger filters, the following alternative definition of an ultrafilter is a key tool in the construction of a nonprincipal ultrafilter.

Theorem 3.8. $\mathscr{F}$ is an ultrafilter if and only if it is a maximal filter.

Proof. Let $\mathscr{F}$ be an ultrafilter, and suppose $\mathscr{G}$ is another filter larger than $\mathscr{F}$. There cannot be any $A \in \mathscr{G} \setminus \mathscr{F}$, since that would imply that $I \setminus A \in \mathscr{F}$ and so $A \cap (I \setminus A) = \emptyset \in \mathscr{G}$.

Suppose that $\mathscr{F}$ is not an ultrafilter. Then, there exists some $A \subseteq I$ such that neither A nor $I \setminus A$ is in $\mathscr{F}$. As such, it follows that for all $B \in \mathscr{F}$, $A \cap B \neq \emptyset$. Otherwise, if there was some $B \in \mathscr{F}$ such that $A \cap B = \emptyset$, then $B \subseteq I \setminus A \implies I \setminus A \in \mathscr{F}$. Therefore, $\mathscr{F} \cup \{A\}$ is a set with the finite intersection property so $\mathscr{F}^{\mathscr{F} \cup \{A\}}$ is a filter larger than $\mathscr{F}$. $\square$

This now enables the proof of

Theorem 3.9. Any set with the finite intersection property, $\mathscr{H}$, can be extended to an ultrafilter, denoted $\mathscr{F}_{\mathscr{H}}$.

Proof. Let $\mathscr{D}$ be the set of all filters including $\mathscr{H}$. $\mathscr{D}$ is partially ordered by the subset relation. Zorn's lemma will be applied to show that $\mathscr{D}$ has a maximal element, i.e., an ultrafilter containing $\mathscr{H}$.

Let $\mathscr{C} \subseteq \mathscr{D}$ be a chain. It must be shown that $\mathscr{C}$ has an upper bound in $\mathscr{D}$. Define

$$\mathscr{F}_{\mathscr{C}} = \bigcup_{\mathscr{G} \in \mathscr{C}} \mathscr{G}$$

Then, $\mathscr{F}_{\mathscr{C}}$ is a filter by the following arguments:

1. If $A, B \in \mathscr{F}_{\mathscr{C}}$, then there are some $\mathscr{G}_A, \mathscr{G}_B \in \mathscr{C}$ that contain A and B. Without loss of generality, assume that $\mathscr{G}_A \subseteq \mathscr{G}_B$. It follows that $A \in \mathscr{G}_B$ so $A \cap B \in \mathscr{G}_B$ and so $A \cap B \in \mathscr{F}_{\mathscr{C}}$.

2. If $A \in \mathscr{F}_{\mathscr{C}}$ then there must be a $\mathscr{G} \in \mathscr{C}$ such that $A \in \mathscr{G}$. If $A \subset B$ then $B \in \mathscr{G}$ so that $B \in \mathscr{F}_{\mathscr{C}}$

3. $\emptyset \notin \mathscr{F}_{\mathscr{C}}$, otherwise there must be some $\mathscr{G} \in \mathscr{C}$ for which $\emptyset \in \mathscr{G}$, which is never the case.

Since $\mathscr{H}$ is a subset of every element of $\mathscr{C}$, it follows that $\mathscr{H} \subseteq \mathscr{F}_{\mathscr{C}}$, and thus $\mathscr{F}_{\mathscr{C}} \in \mathscr{D}$. Further, every element of $\mathscr{C}$ is contained in $\mathscr{F}_{\mathscr{C}}$ so thus $\mathscr{C}$ is an upper bound of the chain.

Therefore, since every chain in $\mathscr{D}$ has an upper bound in $\mathscr{D}$ and Zorn's lemma can be applied so that $\mathscr{D}$ has a maximum element. By Theorem 1.XX, this maximum element is an ultrafilter which contains $\mathscr{H}$. □

Though this theorem is a powerful tool for creating ultrafilters, it makes no guarantee about them being nonprincipal. Consider $\mathscr{H} = \{i\}$ for some $i \in I$. Evidently, $\mathscr{H}$ has the finite intersection property, but $\mathscr{F}_{\mathscr{H}}$ contains this singleton so is principal. Fortunately, it is possible to impose the following restrictions on $\mathscr{H}$ to ensure that $\mathscr{F}_{\mathscr{H}}$ is nonprincipal.

Lemma 3.1. *Let $\mathscr{H} \subseteq \mathscr{P}(I)$ have the finite intersection property. If $\bigcap_{H \in \mathscr{H}} H = \emptyset$ then $\mathscr{F}_{\mathscr{H}}$ is nonprincipal.*

Proof. Proof by contrapositive. Suppose that $\mathscr{F}_{\mathscr{H}}$ is principal so that it includes a singleton $\{i\}$ for some $i \in I$. By the superset property, it follows that $i \in A$ for all $A \in \mathscr{F}_{\mathscr{H}}$. Namely, since $\mathscr{H} \subseteq \mathscr{F}_{\mathscr{H}}$, $i \in H$ for all $H \in \mathscr{H}$. Thus, the intersection of all sets in $\mathscr{H}$ is nonempty. □

If the intersection of a finite number of sets in $\mathscr{H}$ is nonempty while the intersection of every set in $\mathscr{H}$ is empty, it follows that $\mathscr{H}$ must be infinite. Therefore, to demonstrate that a nonprincipal ultrafilter exists, it suffices to show that such a $\mathscr{H}$ can be constructed.

Lemma 3.2. *Let $\mathscr{H} \subseteq \mathscr{P}(\mathrm{I})$ where $\bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H}$ is infinite. Define $\overline{\mathscr{H}} = \mathscr{H} \cup \mathscr{H}^-$ where*

$$\mathscr{H}^- = \{\mathrm{I} \setminus \{i\} \mid \forall \mathrm{H} \in \mathscr{H}, i \in \mathrm{H}\}$$

Then $\overline{\mathscr{H}}$ has the finite intersection property and $\bigcap_{\mathrm{H}\in\overline{\mathscr{H}}} \mathrm{H} = \emptyset$.

Proof. By Lemma 3.XX, it suffices to show that $\overline{\mathscr{H}}$ has the finite intersection property and that the intersection of all of its members is empty.

For the finite intersection property, since $\overline{\mathscr{H}} = \mathscr{H} \cup \mathscr{H}^-$, it suffices to show that a finite number of sets from $\mathscr{H}$ intersected with a finite number of sets from $\mathscr{H}^-$ is nonempty. Observe that $\bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H}$ is a subset of any finite intersection from $\mathscr{H}$, so the problem is once more reduced to demonstrating that $\bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H}$ intersected with a finite number of sets from $\mathscr{H}^-$ is nonempty.

Indeed, let $\mathrm{B}_1, \ldots, \mathrm{B}_n \in \mathscr{H}^-$ for some $n \in \mathbb{N}$. By definition, $\mathrm{B}_k = \mathrm{I} \setminus \{i_k\}$ for all $1 \leq k \leq n$, where $i \in \bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H}$. Thus,

$$\mathrm{B}_1 \cap \cdots \cap \mathrm{B}_n = (\mathrm{I} \setminus \{i_1\}) \cap \cdots \cap (\mathrm{I} \setminus \{i_n\}) = \mathrm{I} \setminus \{i_1, \ldots, i_n\}$$

Therefore,

$$\bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H} \cap (\mathrm{B}_1 \cap \cdots \cap \mathrm{B}_n) = \bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H} \cap (\mathrm{I} \setminus \{i_1, \ldots, i_n\}) = \bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H} \setminus \{i_1, \ldots, i_n\}$$

Since $\bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H}$ is infinite and $\{i_1, \ldots, i_n\}$ is finite, $\bigcap_{\mathrm{H}\in\mathscr{H}} \mathrm{H} \setminus \{i_1, \ldots, i_n\}$ is nonempty. Thus, $\overline{\mathscr{H}}$ has the finite intersection property.

Similarly, the intersection of all elements in $\overline{\mathscr{H}}$ is empty. Since $\overline{\mathscr{H}} = \mathscr{H} \cup \mathscr{H}^-$, it follows that

$$\bigcap_{\mathrm{H}\in\overline{\mathscr{H}}} \mathrm{H} = \bigcap_{\mathrm{K}\in\mathscr{H}} \mathrm{K} \cap \bigcap_{\mathrm{L}\in\mathscr{H}^-} \mathrm{L}$$

So that any $i \in \bigcap_{\mathrm{H}\in\overline{\mathscr{H}}} \mathrm{H}$ must also be in every $\mathrm{K} \in \mathscr{H}$. This implies that $\mathrm{I} \setminus \{i\} \in \mathscr{H}^-$ and so $\mathrm{I} \setminus \{i\} \in \bigcap_{\mathrm{H}\in\overline{\mathscr{H}}} \mathrm{H}$. It follows that this intersection is empty. □

All that now remains to guarantee the existence of a nonprincipal ultrafilter is to find a family of sets which has an infinite intersection, which is a simple task.

Corollary 3.3. *There exists a nonprincipal ultrafilter on any infinite set* I.

Proof. Let $\mathscr{H} = \{\mathrm{I}\}$. Then, the intersection of all elements in $\mathscr{H}$ is evidently I, which is infinite. By Lemma.XX, $\overline{\mathscr{H}}$ has the finite intersection property and an empty intersection of all of its elements. Thus, by Lemma.XX, $\mathscr{F}_{\overline{\mathscr{H}}}$ is a nonprincipal ultrafilter. □

To summarize, an ultrafilter was required to extend equivalence and ordering relations from X to *X, and a nonprincipal ultrafilter was required to avoid the case when $\mathscr{F}$ contains a singleton which guarantees that $^*\mathbb{X} \simeq \mathbb{X}$. However, the two objectives outlined at the beginning of this section still must be demonstrated. First, we must complete the extension of the operations on X to *X.

Definition 3.10. For $f,g \in \mathrm{X}^{\mathrm{I}}$, any extended operation was defined as

$$(f \star^{\mathrm{I}} g)(i) = f(i) \star g(i)$$

This operation is naturally extended to ${}^*\mathrm{X} = \mathrm{X}^{\mathrm{I}} / {}^*\mathrm{E}$

$$[f] \star [g] = [f \star^{\mathrm{I}} g]$$

for $[f],[g] \in {}^*\mathrm{X}$.

Lemma 3.4. $\star^{\mathrm{I}}$ *and* ${}^*\mathrm{T}$ *are well-defined with respect to* ${}^*\mathrm{E}$.

Proof. Let $f, f', g, g' \in \mathrm{X}^{\mathrm{I}}$ so that $f\,{}^*\mathrm{E} f'$ and $g\,{}^*\mathrm{E} g'$. It follows that $[\![f \mathrm{E} f']\!] \in \mathscr{F}$ and $[\![g \mathrm{E} g']\!] \in \mathscr{F}$. Since $f(i)\mathrm{E}f'(i)$ and $g(i)\mathrm{E}g'(i)$ imply that $f(i) \star g(i) \mathrm{E} f'(i) \star g'(i)$,

$$[\![f \mathrm{E} f']\!] \cap [\![g \mathrm{E} g']\!] \subseteq [\![f \star^{\mathrm{I}} g \mathrm{E} f' \star g']\!] \in \mathscr{F}$$

So thus $f \star^{\mathrm{I}} g\,{}^*\mathrm{E} f' \star g'$ Next, assume $f\,{}^*\mathrm{T} g$ so that $[\![f \mathrm{T} g]\!] \in \mathscr{F}$. Similarly, since $f(i)\mathrm{E}f'(i)$, $g(i)\mathrm{E}g'(i)$, and $f(i)\mathrm{T}g(i)$ imply that $f'(i)\mathrm{T}g'(i)$,

$$[\![f \mathrm{E} f']\!] \cap [\![g \mathrm{E} g']\!] \cap [\![f \mathrm{T} g]\!] \subseteq [\![f'\,{}^*\mathrm{T} g']\!] \in \mathscr{F}$$

Therefore, $f'\mathrm{T}g'$.

As such, any binary operation on X^{I} and extended ordering on X^{I} are well-defined. □

With the operation and ordering now established on ${}^*\mathrm{X}$, we can prove that ${}^*\mathbb{X}$ meets the two objectives outlined at the beginning of this section.

Theorem 3.11. Any key operational (associative, commutative, distributive), existential (identities, inverses), or relational (total ordering compatible with operations) properties true in $\mathbb{X}$ are also true in ${}^*\mathbb{X}$

Proof. Let $[f],[g],[h] \in {}^*\mathrm{X}$

1. Associative. Assume that $\star$ is associative in $\mathbb{X}$. Then,

$$\begin{aligned}[f] \star ([g] \star [h]) &= [f] \star [g \star^{\mathrm{I}} h] \\ &= [f \star^{\mathrm{I}} (g \star^{\mathrm{I}} h)] \\ &= [(f \star^{\mathrm{I}} g) \star^{\mathrm{I}} h] \\ &= [f \star^{\mathrm{I}} g] \star [h] \\ &= ([f] \star [g]) \star [h]\end{aligned}$$

2. Commutative. Similarly, assume that $\star$ is commutative in $\mathbb{X}$. It follows that

$$\begin{aligned}[f] \star [g] &= [f \star^{\mathrm{I}} g] \\ &= [g \star^{\mathrm{I}} f] \\ &= [g] \star [f]\end{aligned}$$

3. Distributive. Assume that $\bullet$ is another operation on X that distributes over $\star$. Only the case for left-distributivity will be shown here.

$$\begin{aligned}[f] \bullet ([g] \star [h]) &= [f] \bullet [g \star^{\mathrm{I}} h] \\ &= [f \bullet^{\mathrm{I}} (g \star^{\mathrm{I}} h)] \\ &= [(f \bullet^{\mathrm{I}} g) \star^{\mathrm{I}} (f \bullet^{\mathrm{I}} h)] \\ &= [f \bullet^{\mathrm{I}} g] \star [f \bullet^{\mathrm{I}} h] \\ &= [f] \bullet [g] \star [f] \bullet [h]\end{aligned}$$

4. Existence of identity. Let $e \in \mathrm{X}$ be the identity with respect to $\star$. Then, $[\langle e \rangle_{i \in \mathrm{I}}]$ is the identity of ${}^*\mathrm{X}$ since

$$\begin{aligned}[f] \star [\langle e \rangle_{i \in \mathrm{I}}] &= [f \star^{\mathrm{I}} \langle e \rangle_{i \in \mathrm{I}}] \\ &= [\langle f(i) \star e \rangle_{i \in \mathrm{I}}] \\ &= [\langle f(i) \rangle_{i \in \mathrm{I}}] \\ &= [f]\end{aligned}$$

5. Existence of inverse. For any $x \in \mathrm{X}$, let the inverse with respect to $\star$ of x be x^{-1}. Then, the inverse of $[f]$ is $[\langle f(i)^{-1} \rangle_{i \in \mathrm{I}}]$ since

$$\begin{aligned}[f] \star [\langle f(i)^{-1} \rangle_{i \in \mathrm{I}}] &= [f \star^{\mathrm{I}} \langle f(i)^{-1} \rangle_{i \in \mathrm{I}}] \\ &= [\langle f(i) \star f(i)^{-1} \rangle_{i \in \mathrm{I}}] \\ &= [\langle e \rangle_{i \in \mathrm{I}}]\end{aligned}$$

which is the identity of ${}^*\mathrm{X}$ with respect to $\star$.

6. Operations are compatible with ordering. Suppose that $[f]\mathrm{T}[g]$. Then,

$$\begin{aligned}[f]\mathrm{T}[g] &\iff [\![f^* \mathrm{T} g]\!] \in \mathscr{F} \\ &\implies [\![f \star h^* \mathrm{T} g \star h]\!] \in \mathscr{F} \\ &\iff [f \star h]\mathrm{T}[g \star h] \\ &\iff [f] \star [h]\mathrm{T}[g] \star [h]\end{aligned}$$

□

Theorem 3.12. The map $\phi : \mathbb{X} \to {}^*\mathbb{X}$ given by $\phi(x) = [\langle x \rangle]$ is an injective homomorphism.

Proof. Let $x, y \in \mathrm{X}$.

$$\begin{aligned}\phi(x \star y) &= [\langle x \star y \rangle_{i \in \mathrm{I}}] \\ &= [\langle x \rangle_{i \in \mathrm{I}} \star^{\mathrm{I}} \langle y \rangle_{i \in \mathrm{I}}] \\ &= [\langle x \rangle_{i \in \mathrm{I}}] \star [\langle y \rangle_{i \in \mathrm{I}}] \\ &= \phi(x) \star \phi(y)\end{aligned}$$

Therefore, ϕ is a homomorphism.

Further, assume that $\phi(x) = \phi(y)$. Then,

$$\begin{aligned}\phi(x)^* = \phi(y) &\iff [\langle x \rangle_{i \in \mathrm{I}}] = [\langle y \rangle_{i \in \mathrm{I}}] \\ &\iff \langle x \rangle =^{\mathrm{I}} \langle y \rangle \\ &\iff \{i \in \mathrm{I} \mid x = y\} \in \mathscr{F} \\ &\iff x = y\end{aligned}$$

Thus, ϕ is injective. □

Next, consider the requirements for ${}^*\mathbb{X}$ to contain an unlimited value. Note that X must have no maximum element. Otherwise, the image of that maximum element under ϕ would be greater than or equal to every other element in ${}^*\mathbb{X}$.

Theorem 3.13. Let X have no maximum element. There exists a nonprincipal ultrafilter so that ${}^*\mathrm{X}$ has an unlimited value if and only if there exists a set $\mathrm{Y} \subseteq \mathrm{X}$ such that Y is unbounded in X and $|\mathrm{Y}| \leq |\mathrm{I}|$.

Proof. $\implies$. Let ω be an unlimited value. ω is in the form $[f]$, where $f \in \mathrm{X}^{\mathrm{I}}$. Let $\mathrm{Y} = \{f(i) \mid i \in \mathrm{I}\}$ be the image of f. Note then that for any $x \in \mathrm{X}$, it follows that $\phi(x) < \omega$, so that there exists $i \in \mathrm{I}$ such that $x < f(i)$. Therefore, for any $x \in \mathrm{X}$ there exists an $f(i) \in \mathrm{Y}$ such that $x < f(i)$. Thus, $\mathrm{Y} \subseteq \mathrm{X}$ is unbounded in X. Further, since f is surjective to Y, $|\mathrm{Y}| \leq |\mathrm{I}|$.

$\impliedby$. Let $\mathrm{Y} \subseteq \mathrm{X}$ be unbounded where $|\mathrm{Y}| \leq |\mathrm{I}|$. It follows that there exists a surjective function $f : \mathrm{I} \to \mathrm{Y}$. Since Y is unbounded, for every $x \in \mathrm{X}$ there exists a $y \in \mathrm{Y}$ such that $x < y$. Equivalently, there exists an $i \in \mathrm{I}$ such that $x < f(i)$. With this, each of the sets

$$\mathrm{G}_x = \{i \in \mathrm{I} \mid x < f(i)\}$$

is nonempty. Observe that if $x_1 < x_2$ then $\mathrm{G}_{x_2} \subseteq \mathrm{G}_{x_1}$. Then, let

$$\mathscr{H} = \{\mathrm{G}_x \mid x \in \mathrm{X}\}$$

Observe that $\bigcap_{x\in X} G_x = \emptyset$ since for any $i \in I$, $i \notin G_{f(i)}$ because $f(i) \not< f(i)$. Additionally, $\mathscr{H}$ has the finite intersection property: let $G_{x_1}, \dots, G_{x_n} \in \mathscr{H}$ for some $n \in \mathbb{N}$ where $x_1 < \cdots < x_n$. Then,

$$G_{x_1} \supseteq \cdots \supseteq G_{x_n}$$

so

$$G_{x_1} \cap \cdots \cap G_{x_n} = G_{x_1} \neq \emptyset$$

Therefore, by Lemma 3.XX, $\mathscr{F}_{\mathscr{H}}$ is a nonprincipal ultrafilter. Since $\mathscr{F}_{\mathscr{H}}$ contains $\mathscr{H}$, it follows that $[f]$ is an unlimited value. For any $x \in X$,

$$\begin{aligned} \phi(x) < [f] &\iff [\![\langle x\rangle < f]\!] \in \mathscr{F} \\ &\iff \{i \in I \mid x < f(i)\} \in \mathscr{F} \\ &\iff G_x \in \mathscr{F} \end{aligned}$$

Thus, $[f]$ is an unlimited value. □

Corollary 3.5. *When* $X = \mathbb{N}$ *and* $I = \mathbb{N}$, *every nonprincipal ultrafilter yields unlimited values.*

Proof. Let $\mathscr{F}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Observe that $\mathbb{N} \setminus G_n$ is a finite set for any $n \in \mathbb{N}$. Therefore, $\mathbb{N} \setminus G_n \notin \mathscr{F}$ and so $\mathbb{N} \setminus (\mathbb{N} \setminus G_n) = G_n \in \mathscr{F}$. □

Remark 3.6. The superscripts $\circ^{I}$ and ${}^{*}\circ$ will be omitted on operations and relations in the next sections.

4 Construction

4.1 Hypernaturals

Nearly everything needed to construct the hypernatural numbers has already been completed in the previous section. The first portion of this section will be dedicated to reviewing each relevant definition in the specific case of $X = I = \mathbb{N}$. Beginning with the semiring $(\mathbb{N}, +, \cdot)$ of the natural numbers, the extension to $({}^{*}\mathbb{N}, +, \cdot)$ is made as follows.

Definition 4.1. Let $\mathscr{F}$ be any nonprincipal ultrafilter on $\mathbb{N}$ and let $\mathbb{N}^{\mathbb{N}}$ be the set of all sequences of natural numbers. Then, define ${}^{*}\mathbb{N} = \mathbb{N}^{\mathbb{N}} / \equiv$, where for any $a, b \in \mathbb{N}^{\mathbb{N}}$,

$$a \equiv b \iff [\![a = b]\!] \in \mathscr{F}$$

And similarly,

$$a \leq b \iff [\![a \leq b]\!] \in \mathscr{F}$$

Further, the operations $+$ and $\cdot$ are extended to $\mathbb{N}^{\mathbb{N}}$ and ${}^{*}\mathbb{N}$ with

$$\begin{aligned} [\langle a_n\rangle] + [\langle b_n\rangle] &= [\langle a_n + b_n\rangle] \\ [\langle a_n\rangle] \cdot [\langle b_n\rangle] &= [\langle a_n \cdot b_n\rangle] \end{aligned}$$

It follows that these extended operations and relations are well-defined and form the ordered semiring $({}^*\mathbb{N},+,\cdot)$ with additive identity $[\langle 0\rangle]$ and multiplicative identity $[\langle 1\rangle]$.

As proved more abstractly, $\phi(n) = [\langle n\rangle]$ is an injective homomorphism from $\mathbb{N}$ to ${}^*\mathbb{N}$. Additionally, any unbounded subset of $\mathbb{N}$ yields an unlimited hypernatural number. For example, $\omega = [\langle 1,2,3,\ldots\rangle]$ is unlimited since for any $m \in \mathbb{N}$,

$$\begin{aligned}\phi(m) < \omega &\iff \langle m\rangle < \langle 1,2,3,\ldots\rangle\\ &\iff \{n \in \mathbb{N} \mid m < n\} \in \mathscr{F}\\ &\iff \mathrm{G}_m \in \mathscr{F}\end{aligned}$$

which is true since G_m is cofinite.

Remark 4.1. In later sections, to avoid cumbersome notation as context permits, n will be understood to mean $\phi(n)$ for any $n \in \mathbb{N}$.

Now, there are many interesting properties of the hypernatural numbers not readily visible in as abstract a setting as the last section. As in the naturals, any open interval where each endpoint is a finite distance apart has a finite cardinality. More specifically,

Theorem 4.2. Let $\alpha \in {}^*\mathbb{N}$ and let $n \in \mathbb{N}$. Then, $|(\alpha, \alpha+\phi(m+1))| = m$.

Proof. Proof by induction. Let $\alpha \in {}^*\mathbb{N}$. In the base case of $m = 0$, it must be shown that there exist no hypernatural numbers β such that $\alpha < \beta < \alpha+\phi(1)$. By definition,

$$[\![\alpha < \beta]\!] \cap [\![\beta < \alpha+\phi(1)]\!] = \emptyset \notin \mathscr{F}$$

Thus, $|(\alpha, \alpha+\phi(1))| = 0 = m$.

For the induction step, assume that $|(\alpha, \alpha+\phi(m+1))| = m$. It follows that

$$\begin{aligned}|(\alpha, \alpha+\phi(m+2))| &= |(\alpha, \alpha+\phi(m+1) \cup \{\alpha+\phi(m+1)\} \cup (\alpha+\phi(m+1), \alpha+\phi(m+1)+\phi(1))|\\ &= |(\alpha, \alpha+\phi(m+1)| + |\{\alpha+\phi(m+1)\}| + |(\alpha+\phi(m+1), \alpha+\phi(m+1)+\phi(1))|\\ &= m+1+0\end{aligned}$$

as desired. □

Corollary 4.2. *If* $\alpha \in {}^*\mathbb{N}$ *is finite, then* $\alpha = \phi(n)$ *for some* $n \in \mathbb{N}$.

Proof. Assume that $\alpha \neq \phi(0)$. Since α is finite, there exists a $k \in \mathbb{N}$ such that $\alpha < \phi(k+1)$. As such, $\alpha \in (\phi(0), \phi(k+1))$. It follows that $|(\phi(0), \phi(k+1))| = k$. Observe, however, that for all $n \in (0, k+1)$, $\phi(n) \in (\phi(0), \phi(k+1))$. There are k different n, so thus $\alpha = \phi(n)$ for one of these n. □

With this, the hypernatural numbers can split into sets of unlimited and finite values.

Definition 4.3. Let ${}^*\mathbb{N}_{\mathrm{A}} = \phi(\mathbb{N})$ be the set of all appreciable (or finite) hypernatural numbers. Let ${}^*\mathbb{N}_{\mathrm{U}} = {}^*\mathbb{N} \setminus {}^*\mathbb{N}_{\mathrm{A}}$ be the set of all unlimited hypernaturals.

It follows that $\mathbb{N} \simeq {}^*\mathbb{N}_{\mathrm{A}}$ since ϕ is surjective to its image and so is an isomorphism.

The appreciable and unlimited hypernatural numbers are closed under $+$ and $\cdot$. Evidently, if $\alpha, \beta \in {}^*\mathbb{N}_{\mathrm{U}}$, then $\alpha < \alpha + \beta$ and $\alpha < \alpha \cdot \beta$, so thus $\alpha + \beta, \alpha \cdot \beta \in {}^*\mathbb{N}_{\mathrm{U}}$. Additionally, if $\alpha, \beta \in {}^*\mathbb{N}_{\mathrm{A}}$, then $\alpha = \phi(n)$ and $\beta = \phi(m)$ for some $n, m \in \mathbb{N}$. Thus, $\alpha + \beta = \phi(n+m) \in {}^*\mathbb{N}_{\mathrm{A}}$ and $\alpha \cdot \beta = \phi(n \cdot m) \in {}^*\mathbb{N}_{\mathrm{A}}$.

In the naturals, if $n > m$, then there exists a k such that $n = m + k$. This fact is similarly true for the hypernaturals.

Lemma 4.3. *Let* $\alpha, \beta \in {}^*\mathbb{N}$. *If* $\alpha > \beta$, *then there exists a* $\gamma \in {}^*\mathbb{N}$ *such that* $\alpha = \beta + \gamma$.

Proof. If $\alpha = \beta$, then $\gamma = \phi(0)$. Without loss of generality, assume that $\alpha > \beta$. Then,

$$\alpha > \beta \iff \{n \in \mathbb{N} \mid \alpha_n > \beta_n\} \in \mathscr{F}$$

Whenever $\alpha_n > \beta_n$, then there exists a $c_n \in \mathbb{N}$ such that $\alpha_n = \beta_n + c_n$. Then, define $\gamma_n = c_n$ whenever $\alpha_n > \beta_n$ and $\gamma_n = 0$ otherwise, it follows that $\alpha = \beta + \gamma$. □

A property which the hypernatural numbers do not share with the naturals is the least upper bound. That is, there exists a bounded subset of ${}^*\mathbb{N}$ for which there is no minimum upper bound. In particular,

Lemma 4.4. ${}^*\mathbb{N}_{\mathrm{A}}$ *has no least upper bound.*

Proof. Evidently, a hypernatural number is an upper bound of ${}^*\mathbb{N}_{\mathrm{A}}$ if and only if it is unlimited. As such, let $\alpha \in {}^*\mathbb{N}_{\mathrm{U}}$. It follows by Lemma 4.XX that since $\alpha > \phi(1)$, there exists a $\beta \in {}^*\mathbb{N}$ such that $\alpha = \beta + \phi(1)$. It cannot be that β is an appreciable hypernatural number, otherwise α would also be since the appreciables are closed under addition. Thus, β is an unlimited value less than α. Since α was an arbitrary upper bound of ${}^*\mathbb{N}_{\mathrm{A}}$, it follows that ${}^*\mathbb{N}_{\mathrm{A}}$ has no least upper bound. □

To further distinguish the hypernaturals, they contain uncountable bounded open intervals, whereas the naturals only contain finite bounded open intervals.

Theorem 4.4. Let $\alpha, \beta \in {}^*\mathbb{N}$ such that $\alpha < \beta$ and where $\beta \neq \alpha + \phi(n)$ for all $n \in \mathbb{N}$. Then, $|(\alpha, \beta)| \geq 2^{\aleph_0}$.

Proof. Since $\beta > \alpha$, there exists a $\gamma \in {}^*\mathbb{N}$ such that $\beta = \alpha + \gamma$. Since $\beta \neq \alpha + \phi(n)$ for any $n \in \mathbb{N}$, it follows that γ is an unlimited hypernatural number. Let $\psi : (0,1) \subseteq \mathbb{R} \to (\alpha, \beta)$ be defined by $\psi(x) = [\langle \alpha_n + \lfloor x\gamma_n \rfloor \rangle]$. It suffices to prove that $\psi(x)$ is always contained in (α, β) and that ψ is injective.

First, let $x \in (0,1)$. Then,

$$\alpha < \psi(x) < \beta \iff \{n \in \mathbb{N} \mid \alpha_n < \alpha_n + \lfloor x\gamma_n \rfloor\} \in \mathscr{F} \wedge \{n \in \mathbb{N} \mid \alpha_n + \lfloor x\gamma_n \rfloor < \beta_n\rfloor\} \in \mathscr{F}$$

Which is true since $\lfloor x\gamma_n \rfloor > 0$ and $\lfloor x\gamma_n \rfloor < \gamma_n$ for all $x \in (0,1)$ and $n \in \mathbb{N}$.

Next, suppose that for $x, y \in (0,1)$, $x \neq y$. Without loss of generality, assume that $x < y$ and write $y = x + d$. Observe then that

$$\begin{aligned}\psi(x) < \psi(y) &\iff \psi(x) < \psi(x+d)\\ &\iff \{n \in \mathbb{N} \mid \alpha_n + \lfloor x\gamma_n \rfloor < \alpha_n + \lfloor (x+d)\gamma_n \rfloor\} \in \mathscr{F}\\ &\iff \{n \in \mathbb{N} \mid \lfloor x\gamma_n \rfloor < \lfloor x\gamma_n + d\gamma_n \rfloor\} \in \mathscr{F}\end{aligned}$$

Which is true whenever $d\gamma_n \geq 1$. By the Archimedean property, there exists $\mathrm{N} \in \mathbb{N}$ such that $\mathrm{N} \geq \frac{1}{d}$. Then, since γ is unlimited, it follows that $\{n \in \mathbb{N} \mid \gamma_n > \mathrm{N}\} \in \mathscr{F}$. Thus, $\psi(x) < \psi(y)$ so that $\psi(x) \neq \psi(y)$. Therefore, ψ is injective.

Since $|(0,1)| = 2^{\aleph_0}$, it follows that $|(\alpha,\beta)| \geq 2^{\aleph_0}$ □

To conclude this section, one interesting pursuit is the creation of a "hyperhyper-natural" number system. By design, $({}^*\mathbb{N}, +, \cdot)$ is itself an ordered semiring. Therefore, it is possible to apply the constructions in Section 3 once more to create another new number system.

Just as the hypernatural numbers contain values which are unlimited in respect to the naturals, any "hyperhypernatural" structure should have values that are unlimited in respect to the hypernaturals. For this to be true, there must exist an unbounded subset of cardinality less than or equal to that of the indexing set. Thus, the path towards constructing the "hyperhypernaturals" begins by attempting to identify a nontrivial unbounded subset of ${}^*\mathbb{N}$.

Theorem 4.5. If $\mathrm{B} \subseteq {}^*\mathbb{N}$ is unbounded, then $|\mathrm{B}| \geq 2^{\aleph_0}$.

Proof. Proof by contradiction. Suppose that B is a countable unbounded subset of ${}^*\mathbb{N}$. Write B as $\{[s^1], [s^2], \ldots\}$, where $s^1, s^2, \ldots \in \mathbb{N}^{\mathbb{N}}$. It suffices to create a sequence larger than each of the s^n. Let $\mathrm{S} \in \mathbb{N}^{\mathbb{N}}$ where

$$\mathrm{S}_n = \sum_{k=1}^{n} s_n^k$$

Then, for any $m \in \mathbb{N}$, $s^m < \mathrm{S}$ since for all $n \geq m$,

$$\mathrm{S} = s_n^m + \sum_{k=1}^{m-1} s_n^k + \sum_{k=m+1}^{n} s_n^k \geq s_n^m$$

Therefore, [S] is an element of ${}^*\mathbb{N}$ that is greater than every element of B. If [S] is larger than every element in B we are done. Otherwise, if [S] is equal to some element of B, then it must be the maximal element. As such, $[\mathrm{S}] + \phi(1)$ is larger than every element of B. Since B was arbitrary, it follows that no countable unbounded subset of ${}^*\mathbb{N}$ exists.

$$
\begin{array}{c|cccccc}
 & S_1 & S_2 & S_3 & S_4 & S_5 & \\ \hline
s_1 & s_1^1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & \cdots \\
 & & + & + & + & + & \\
s_2 & s_2^1 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & \cdots \\
 & & & + & + & + & \\
s_3 & s_3^1 & s_3^2 & s_3^3 & s_3^4 & s_3^5 & \cdots \\
 & & & & + & + & \\
s_4 & s_4^1 & s_4^2 & s_4^3 & s_4^4 & s_4^5 & \cdots \\
 & & & & & + & \\
s_5 & s_5^1 & s_5^2 & s_5^3 & s_5^4 & s_5^5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

□

Since |I| ≥ |B| for any extension that contains unlimited values, it follows that the "hyperhypernatural" numbers cannot be created using sequences of natural numbers. The most obvious choice, then, is choosing ${}^*\mathbb{N}$ itself as the indexing set.

4.2 Hyperintegers

4.3 Hyperrationals

4.4 Hyperreals

5 Calculus

6 Cardinality

By Lemma 3.XX, the cardinality of the hypernaturals is greater than or equal to the reals. Additionally, since an injective function exists between the sets ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, and ${}^*\mathbb{R}$,

$$2^{\aleph_0} \leq |{}^*\mathbb{N}| \leq |{}^*\mathbb{Z}| \leq |{}^*\mathbb{Q}| \leq |{}^*\mathbb{R}|$$

Since ${}^*\mathbb{R} = {}^*\mathbb{Q}^{\mathbb{N}} / \sim$, it follows that $|{}^*\mathbb{R}| \leq |{}^*\mathbb{Q}^{\mathbb{N}}|$. Thus,

$$2^{\aleph_0} \leq |{}^*\mathbb{R}| \leq (2^{\aleph_0})^{\aleph_0}$$

Which, by cardinal arithmetic[1],

$$2^{\aleph_0} \leq |{}^*\mathbb{R}| \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$$

Therefore, $|\mathbb{R}| = |{}^*\mathbb{N}| = |{}^*\mathbb{Z}| = |{}^*\mathbb{Q}| = |{}^*\mathbb{R}|$

7 Discussion

References

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